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Vibration of a Mistuned Bladed-Disk Assembly Using Structurally Damped Beams

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A model of a mistuned bladed disk is created and then analyzed for both free and forced vibrations. The model consists of structurally damped, Euler–Bernoulli beams connected at one end using springs. The model is useful in the prediction of the response of a turbine engine bladed disk in which only some of the blades have added damping. (This situation arises in the experimental analysis of damping treatments.) The natural frequencies and eigenfunction sets are obtained by assuming harmonic motion and imposing the boundary conditions. The eigensolution was verified using a small NASTRAN model. The forced response due to phased, harmonic tip loads is obtained by modalizing the equations using a technique recently developed by Turcotte (Turcotte, J. S., "An Analytical Model for the Forced Response of Cyclic Structures," *Journal of Sound and Vibration*, Vol. 209, No. 3, 1998, pp. 531–536). The physical response is obtained as a function of forcing frequency by truncation and summation of the modal responses. The results are in general agreement with those of existing discrete models but have much greater detail.

Nomenclature

A_i, B_i, C_i, D_i	= eigenfunction coefficients of the general solution
b_i	= thickness of beam i
$(EI)_i$	= flexural rigidity of beam i
F	= magnitude of the applied point loads
h	= height of all beams
j	= imaginary unit $\sqrt{-1}$
k_{lc}	= linear coupling spring constant
k_{lg}	= linear grounding spring constant
k_{rc}	= rotational coupling spring constant
k_{rg}	= rotational grounding spring constant
l	= length of all beams
m_i	= mass per unit length of beam i
n	= number of beams, range of index i
p_i	= distributed transverse load applied to beam i
x	= independent spatial variable
Y_{is}	= eigenfunction of beam i in mode s
y_i	= displacement of beam i
β_i	= eigenvalue of mode i
γ_i	= structural damping factor of beam i
δ	= Dirac's delta function
η_r	= time-dependent modal coordinate of mode r
θ_i	= phase of load applied to beam i compared to beam 1
ρ	= density of beam material
ω	= excitation frequency
$\bar{\omega}_r^2$	= eigenvalue of mode r

Introduction

INTEREST in the study of mistuned bladed disks is accelerating due to continually increasing performance requirements in military aircraft turbine engines. Mistuning is a term used for the introduction of imperfections into a structure that usually cause a qualitative change in the eigenstructure of an otherwise symmetric structure. The early work of Ewins[1] used receptance methods to connect beams to a disk but did not include damping. In a more recent study, Shankar and Keane[2] used both finite element and receptance methods to predict the vibrational energy flow between two beams connected by springs. Receptance methods use the summation of eigenfunctions of components (Green functions) as a basis for approximating the eigenfunctions of the system.

Because of the practice of testing damped blades by applying damping treatments to only a few blades of an otherwise undamped assembly, Hollkamp et al.[3] studied the response of a lumped model of a mistuned bladed-disk assembly with unequal damping. They used single-mass elements for each blade and used the strain energy method to determine the modal damping rather than performing a complex eigenanalysis. Their results indicate that the level of coupling between the blades plays a major role in one's ability to discern the contribution of damping treatments. More specifically, moderate coupling combined with small mistuning can cause modal strain energy to be distributed among damped and undamped blades. This makes it difficult to determine the effectiveness of damping treatments.

The present work refines the analysis in Ref. 3 by using beams to represent the blades and by performing the complex eigenanalysis. In addition, we use the approach outlined by Turcotte,[4] which does not rely on Green functions of the components and, therefore, yields exact results for the eigenvalues and eigenfunction sets of the system. We investigate both lightly and moderately coupled systems and compare the results.

Euler–Bernoulli Beam Model

An illustration of the model used in this research is shown in Fig. 1. The blades are mounted symmetrically about a circle and lie in a plane. Vibration is restricted to the same plane. At the hub end, each beam rotates against a grounded torsional spring k_{rg} and translates against a grounded linear spring k_{lg}. The beams are connected by rotational coupling springs k_{rc} and linear coupling springs k_{lc}. The beams are free at the other end. The displacement of beam i is designated $y_i(x, t)$, and the independent variable x is not distinguished among the beams. As Euler–Bernoulli beam theory assumes small, linear deformations, the rotation of the torsional springs is assumed equal to the slope of the beam displacement at the hub, $\partial y_i/\partial x(0, t)$. Additionally, the angle the linear coupling springs k_{lc} makes with the beams is assumed to be a right angle, as proper accounting of the force in the spring would only involve multiplication by a common angle sine, which could be incorporated into the spring constant k_{lc}.

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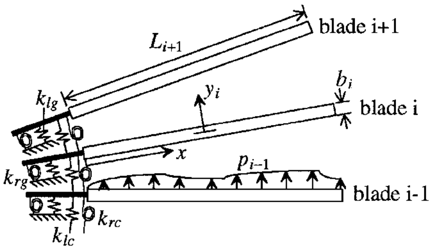

Fig. 1 Idealized bladed-disk assembly for an arbitrary number of blades.

Application of Hamilton's principle for uniform beams yields the equations of motion

$$(1 + j\gamma_i)(EI)_i \frac{\partial^4 y_i}{\partial x^4} + m_i \frac{\partial^2 y_i}{\partial t^2} = p_i(x, t) \tag{1}$$

where $m_i = \rho h b_i$, and the boundary conditions

$$\begin{aligned} 0 &= (1 + j\gamma_i)(EI)_i \frac{\partial^2 y_i}{\partial x^2}(0, t) - k_{\text{rg}} \frac{\partial y_i}{\partial x}(0, t) \\ &\quad + k_{\text{rc}}\left[\frac{\partial y_{i+1}}{\partial x}(0, t) + \frac{\partial y_{i-1}}{\partial x}(0, t) - 2\frac{\partial y_i}{\partial x}(0, t)\right] \\ 0 &= (1 + j\gamma_i)(EI)_i \frac{\partial^3 y_i}{\partial x^3}(0, t) + k_{\text{lg}} y_i(0, t) \\ &\quad + k_{\text{lc}}[2y_i(0, t) - y_{i+1}(0, t) - y_{i-1}(0, t)] \\ 0 &= \frac{\partial^2 y_i}{\partial x^2}(l, t) = \frac{\partial^3 y_i}{\partial x^3}(l, t) \end{aligned} \tag{2}$$

It can be easily verified that these equations are self-adjoint according to the definition given in Ref. 4 even for a mistuned assembly, provided the mistuning arises from parameter differences among the beams. (If the mistuning is the result of some other irregularities, the problem is more difficult to solve, but the response is not qualitatively different.) Thus, the system will uncouple due to the orthogonality of the eigenfunction sets according to the modalization procedure given in Ref. 4.

In the specific analysis performed, eight uniform rectangular steel ($E = 220.6$ GPa and $\rho = 7500\ \text{kg/m}^3$) blades of dimensions $l = 0.5$, $h = 0.02$, and $b = 0.01$ m were used. The blades were mistuned by alteration of the blade thickness b from the nominal value using a random number generator with a 1% standard deviation of a Gaussian distribution (actual blade thicknesses are indicated in Fig. 2). Blades 6–8 were moderately damped ($\gamma = 0.02$), and the other five blades were lightly damped ($\gamma = 0.002$). A phased harmonic tip load was applied to the beams:

$$p_i = F\delta(x - l)\exp[j(\omega t + \theta_i)], \qquad \theta_i = \frac{2\pi(i - 1)}{n} \tag{3}$$

where F is a constant. These parameters were chosen to parallel the analysis in Ref. 3.

Eigenanalysis

The eigenanalysis was performed in the usual way by assuming harmonic motion (the free response is not really harmonic, but the decay is accounted for in the imaginary part of $\bar{\omega}$) and separation of variables:

$$\begin{aligned} y_i(x, t) &= Y_i(x)g(t) = (A_i \cos\beta_i x + B_i \sin\beta_i x \\ &\quad + C_i \cosh\beta_i x + D_i \sinh\beta_i x)\, e^{j\bar{\omega}t} \end{aligned} \tag{4}$$

where

$$\beta_i^4 = \frac{m_i \bar{\omega}^2}{(EI)_i(1 + j\gamma_i)} \tag{5}$$

As there are four boundary conditions for each beam, the eigenvalue problem consists of a $4n \times 4n$ ($n = 8$ in our case) matrix whose

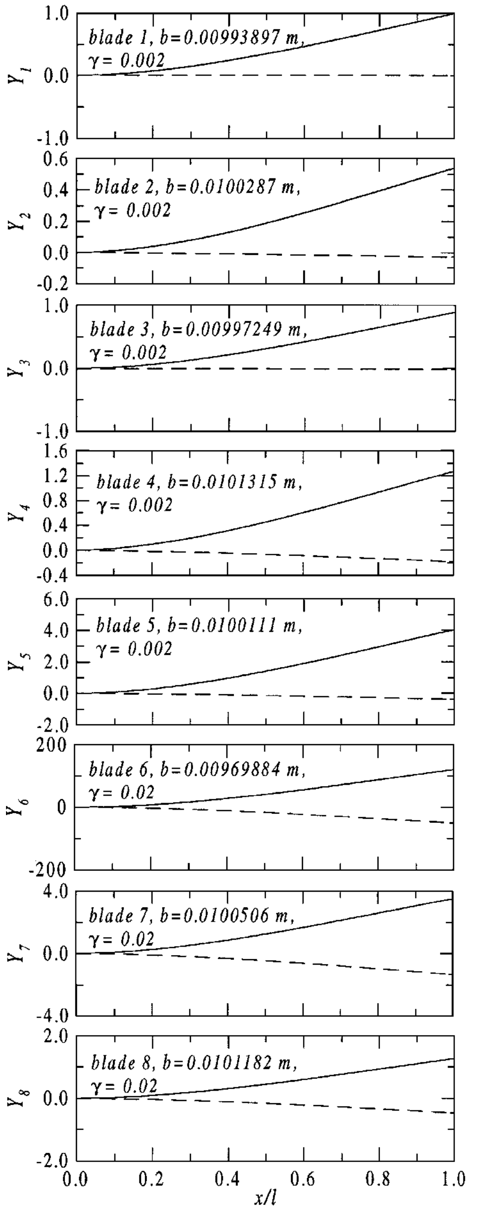

Fig. 2 Eigenfunctions of mode 1 for moderate coupling: ——, real part, and – – –, imaginary part.

determinant will vanish when $\bar{\omega}$ coincides with an eigenvalue. This must generally be done by iteration or a complex root solver until the desired number of complex eigenvalues is found. The $4n$ eigenfunction coefficients A_i–D_i are then determined by assuming a value for one of them, moving the corresponding column to the other side of the equation, substituting the desired eigenvalue, deleting a row from the rank deficient boundary condition matrix, and solving for the remaining $4n - 1$ coefficients.

Eigenanalysis was performed on the eight-bladed model for two cases of blade-to-blade coupling. A precision of 120 decimal places was used to calculate the characteristic determinant (of the boundary condition matrix). The eigenvalues were found to 12 decimal places, and this precision was propagated through the remaining analysis using a machine precision of 16 decimal places. The first case, considered light coupling, had ratios of the coupling and grounding springs of $k_{\text{rc}}/k_{\text{rg}} = k_{\text{lc}}/k_{\text{lg}} = \frac{1}{30}$. Modes computed for this case had amplitudes localized to a single blade, i.e., the motion of blades was not coupled. The eight first bending modes are ranked according to the beam thickness: The first mode has a large response in the thinnest blade and so on.

A second case was analyzed to study the effect of moderate coupling ($k_{\text{rc}}/k_{\text{rg}} = k_{\text{lc}}/k_{\text{lg}} = 1$). The group of first bending modes for this case includes both localized and coupled behavior. The first mode shape for the moderately coupled case is shown in Fig. 2,

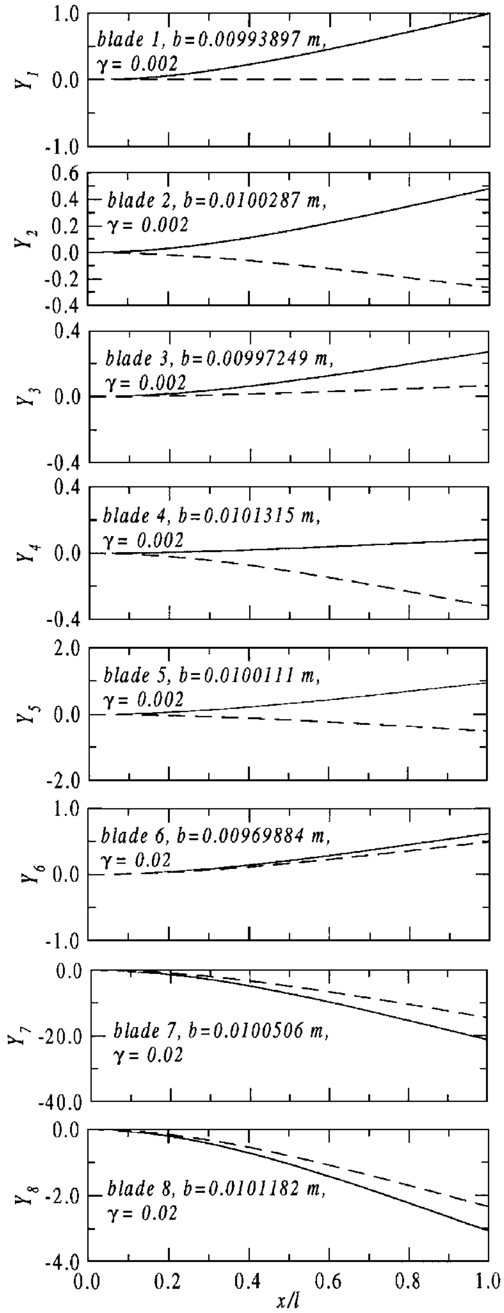

Fig. 3 Eigenfunctions of mode 6 for moderate coupling: ——, real part, and - - -, imaginary part.

where the real parts of the eigenfunctions are shown by solid lines and the imaginary parts by dashed lines. This mode is localized to blade 6, a moderately damped blade. Blade 5, the blade with the next highest amplitude for this mode, has an amplitude of only 3% of that of blade 6. In contrast to the first mode, mode 6 exhibits coupling, as shown in Fig. 3. Blade 7, a moderately damped blade, has the highest amplitude, but blade 8 also has significant amplitude (15%). Other modes in the first bending group also exhibit coupling, but none of the modes have significant coupling between damped and undamped blades. If a mode has coupling of damped and undamped blades, it is difficult to identify the damping contribution of a single damped blade.

We should reemphasize here that this is an exact result within the confines of structurally damped Euler–Bernoulli beam theory. No expansion of component modes has been assumed.

A two-blade model was executed on NASTRAN using 80 structurally damped CBARS and eight CELAS1 elements to verify the analytical eigenanalysis. The NASTRAN results matched the analytical results within a fraction of a percent on eigenvalues, mode shapes, and phasing.

Forced Response

To modalize the equations following the procedure given in Ref. 4, we first expand the beam responses in an infinite series of their eigenfunctions (the ones from the exact eigenanalysis, not the eigenfunctions of a single-component beam):

$$y_i(x, y, t) = \sum_{r=1}^{\infty} \eta_r(t) Y_{ir}(x) \tag{6}$$

These eight expansions are substituted into their corresponding component equations of motion (1).

Next, these n equations are multiplied by the eigenfunction for their respective components of an arbitrary mode. The resulting products are then integrated over the domain of the components. If we also assume harmonic time dependence at frequency ω, as we should expect from the harmonic applied load, these equations may be written as

$$\begin{aligned}
&\int_0^l (1 + j\gamma_i)(EI)_i \frac{\partial^4}{\partial x^4}\left[\sum_{r=1}^{\infty} \eta_r(t) Y_{ir}\right] Y_{is}\,\mathrm{d}x \\
&\quad + \int_0^l m_i \left[\sum_{r=1}^{\infty} \ddot{\eta}_r(t) Y_{ir}\right] Y_{is}\,\mathrm{d}x = \int_0^l p_i Y_{is}\,\mathrm{d}x \\
&\quad = F Y_{is}(l) \exp[j(\omega t + \theta_i)]
\end{aligned} \tag{7}$$

We emphasize that, at this point, all terms in the series may survive the integration, as the individual eigenfunctions of a given beam and mode need not be orthogonal to any other eigenfunction of that beam. Indeed, it should be clear from Fig. 2 that the eigenfunctions of a given beam will be very similar for any group of modes, such as the first bending modes. Therefore, up to n terms in the preceding summations (after integration) will typically be comparable in size, whereas the remaining terms may be small in comparison. It is the summation of all n Eqs. (7) that yields the modal equations of motion. Thus, the final step in modalizing the equations is

$$\begin{aligned}
&\sum_{i=1}^{n} \left\{ \int_0^l \frac{(1 + j\gamma_i)(EI)_i}{m_i} \frac{\partial^4}{\partial x^4}\left[\sum_{r=1}^{\infty} \eta_r(t) Y_{ir}\right] Y_{is}\,\mathrm{d}x \right. \\
&\quad \left. + \int_0^l \left[\sum_{r=1}^{\infty} \ddot{\eta}_r(t) Y_{ir}\right] Y_{is}\,\mathrm{d}x = \frac{F Y_{is}(l)}{m_i} \exp[j(\omega t + \theta_i)] \right\}
\end{aligned} \tag{8}$$

where we have also divided each equation by its constant m_i. Because the eigenfunctions are of the form (4), taking the four derivatives in the first term of Eq. (8) gives back the same eigenfunction with the coefficient β_i^4, which is given by Eq. (5). It should be clear from Eq. (8) that, by substitution from Eq. (5), the same series terms for all n equations will have the same coefficient $\bar{\omega}_r^2$:

$$\begin{aligned}
&\sum_{i=1}^{n} \left\{ \sum_{r=1}^{\infty} \left[\bar{\omega}_r^2 \eta_r(t) + \ddot{\eta}_r(t)\right] \int_0^l Y_{ir} Y_{is}\,\mathrm{d}x \right. \\
&\quad \left. = \frac{F Y_{is}(l)}{m_i} \exp[j(\omega t + \theta_i)] \right\}
\end{aligned} \tag{9}$$

None of the integrals need vanish; however, the sum over n of these integrals for each mode r will vanish except for the s term because the problem is self-adjoint according to the definition given in Ref. 4. The decoupled modal equations are then

$$\begin{aligned}
&\bar{\omega}_s^2 \eta_s(t) + \ddot{\eta}_s(t) \\
&\quad = \left\{ F \sum_{i=1}^{n} Y_{is}(l) \exp[j(\omega t + \theta_i)] \right\} \Bigg/ \left(\sum_{i=1}^{n} m_i \int_0^l Y_{is}^2\,\mathrm{d}x \right)
\end{aligned} \tag{10}$$

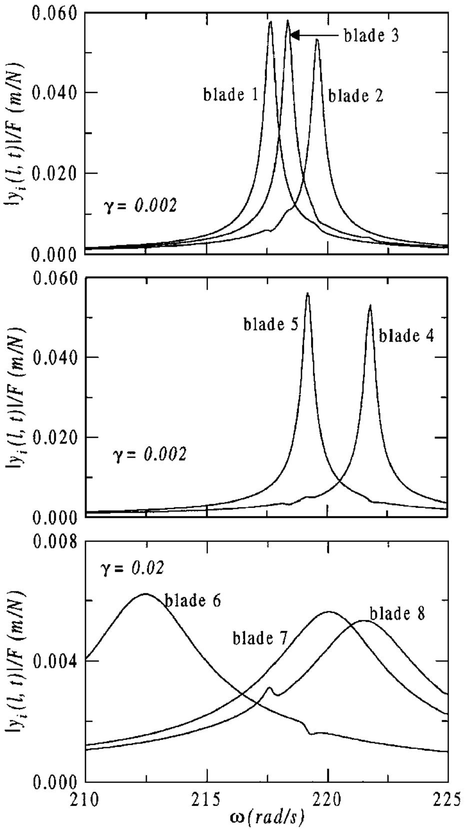

Fig. 4 Magnitude of the blade tip forced response due to phased tip loads (lightly coupled system).

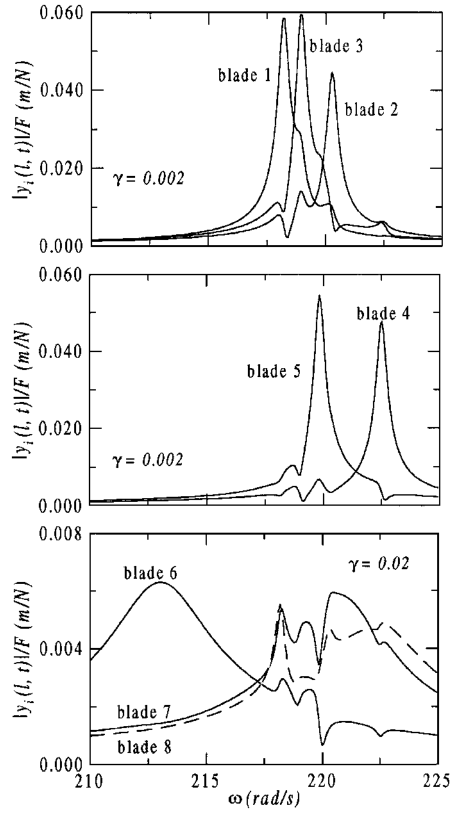

Fig. 5 Magnitude of the blade tip forced response due to phased tip loads (moderately coupled system).

The modal equations of motion are easily solved by assuming a particular solution of exponential form, but there are infinitely many of them. According to the usual procedure in modal analysis, the modal response summation (6) is truncated after inclusion of all modes having frequencies below a multiple (for example, three) of the driving frequency. For our study, we assume the driving frequency is near the first modal frequency and include in the response all modes through the third bending group (the first 24 modes, which include all natural frequencies below 7000 rad/s), although the error in using only the first eight modes is less than 10% as compared with that using 24 modes. Figure 4 plots the forced response magnitude of the blade tips [we set $x = l$ in Eq. (6), sum the first 24 terms, and then take the magnitude] through the first bending frequency range for the case of light coupling. Figure 4 shows that, when coupling is light and modes are localized, the forced response for a blade is dominated by a single mode. In this case, damping can be easily estimated. In Fig. 5, we plot the same results for the moderately coupled case. It should be clear from these plots that moderate coupling can complicate the identification of modal damping.

We also analyzed the case in which the loads are in phase ($\theta_i = 0$). In this case, the peak amplitudes change slightly, but there is no qualitative change in the response; therefore, we do not present these results.

Conclusions

The results show that, depending on the amount of coupling between the blades, the response is multimodal, making it difficult to extrapolate the results of an experiment run on a partially damped assembly to that of a fully damped assembly. The results are in general agreement with those of existing models but obviously include higher-order modes and highly detailed mode shapes. By including the complex eigenanalysis, this work also provides more accurate forced response results and detailed mode shape information, as the phasing of the mode shapes is included.

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